



Application of Branching Process Modelling to Solve Real Life Challenges

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ABSTRACT

Branching process theory or Galton–Watson process provides appropriate mathematical models to describe the probabilistic evolution of systems whose components (cells, particles or individuals in general) reproduce and die after a life period. They are used to describe random systems such as chain reactions, survival of family names, pest eradication, population development and gene propagation. In this work the probability of extinction; the time of extinction and the probability of total progeny is demonstrated with clear illustrations. The decay of the families of men who occupied conspicuous positions in past times has been a subject of frequent remark, and has given rise to various conjectures. The branching process is a very useful tool in many real-life problems (non-deterministic problems) and random phenomena. It has been shown that if the mean number of offspring per individual is more than 1 (that is, on average, if individuals replace themselves with a bit extra), then the branching process is not guaranteed to die out. However, if the mean number of offspring per individual is 1 or less, the process is guaranteed to become extinct.

Keywords: Branching process, size of population, population-dependence, extinction time

INTRODUCTION

The branching process is a simple but elegant model of population growth. It is also called the Galton-Watson Process; because some of the early theoretical results about the process derive from the correspondence between Sir Francis Galton and Reverend Henry William Watson, Klebaner (1985). Branching process theory provides appropriate mathematical models to describe the probabilistic evolution of systems whose components (cells, particles, individuals in general) reproduce and die after a life period. (Becker, 1989; Farrington & Grant, 1999; Haccou, Jagers & Vatutin, 2005; Diekmann & Heesterbeek, 2000; Lyons & Peres, 2016). A branching process is one where an initial number of particles create more particles of the same or different type and these particles then create other particles with the system developing according to some probability laws. In a branching process time is measured discretely, particles are considered to behave independently and there exists a certain probability of producing new particles (Bagley, 1986).

The decay of the families of men who occupied conspicuous positions in past times has been a subject of frequent remark, and has given rise to various conjectures (Gonzalez & Molina, 1998). There are objects that can generate

additional objects of the same kind; these could be men or bacteria reproduced by familiar biological methods or neutrons in a chain reaction. An initial set of objects which we call the 0th generation have children that are called the first generation: their children are the second generation and so on. The process is affected by chance events. In this work, we choose the simplest possible mathematical description of such a situation corresponding to the model of Galton and Watson (Smith & Wahl, 2005; Lyons & Peres, 2016).

According to Galton (1891) in his book “Hereditary genius” he treated the different social groups (such as nobility and judges), which he ascribed the biological reasons causing reduced fertility. Their law of population is such that in each generation a_0 percent of adult males have no male children who reach adult life; a_1 have one such male child, a_2 have two; and so on up to a_5 who have five children. What proportion of surnames would have become extinct after r generations? How many instances

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would there be of the surnames being held by m persons? The statistical nature of the problem was unclear and they tried to work out some numerical examples. The appearance of De Candolle's book however caused Galton to revisit the problem of the extinction of families and to formulate it differently. The disappearance of family names and extinction of families are of course just two different ways to describe the same thing.

Galton thought to himself that one could compute "using the laws of probability" the percentage of family that dies out and that consequently, one should be able to determine whether or a given number of family died out because of reduced fertility (Albertsen, 1995). Citing historical examples of Galton-Watson process is complicated due to the history of family names often deviating significantly from the theoretical model. Notably, new names can be created, existing names can be changed over a person's lifetime, and people historically have often assumed names of unrelated persons, particularly nobility. Thus, a small number of family names at present is not in itself evidence for names having become extinct over time, or that they did so due to dying out of family name lines – that requires that there were more names in the past and that they die out due to the line dying out, rather than the name changing for other reasons.

In the concept of branching theory, first we keep track of the size of the successive generation not the time which individual objects are born or their individual family relationships. We denote by X_0, X_1, X_2, \dots the number in the 0th, first, second, ... generations (we sometimes can interpret X_0, X_1, X_2, \dots as a size of population at a sequence of point in time). Furthermore, we make the following assumptions

- If the size of the n^{th} generation is known, then the probability law governing the later generations does not depend on the sizes of generation preceding the n^{th} generation. In other words, X_0, X_1, X_2, \dots is a form of Markov chain. We shall always make additional assumption that the transitional probability does not vary with time.
- The Markov chain considered in this context have a very special property corresponding

to the assumption that different objects do not interfere with one another: the number of children born to an object does not depend on how many objects are present.

A typical example of the application of the branching process is in the survival of family names.

Family names are inherited by sons only. And a man produces a random number of male offspring during his life time. These male offspring produces another random number of offspring's in the subsequent generations. Hence these descendants make up what is called branching. Lotka (1931) used the theory of branching process to address a problem of survival of family names. The following is his approach; He assumed a zero modified geometric distribution to fit the offspring distribution of the 1920's American male population in the zero modified geometric distribution, the probability that a father has j sons is:

$$P_j = bp^{j-1}, j = 1, 2, 3, \dots$$

Lotka assumed $b = \frac{1}{5}$ and $p = \frac{1}{2}$. Therefore the probability of having no sons is $P_0 = \frac{1}{2}$, the probability of having one son is $P_1 = \frac{1}{5}$ and so on. Then the offspring probability has the following form

$$\begin{aligned} G(s) &= p_0 + p_1s + p_2s^2 + \dots + \dots \\ &= p_0 + \sum_{j=1}^{\infty} bp^{j-1}s^j = p_0 \\ &\quad + \frac{bs}{1-ps} \\ &= \frac{1}{2} + \frac{s}{5-3s} \end{aligned}$$

The mean of the offspring is $m = g'(1) = \frac{5}{4} > 1$. According to the fundamental case $\mu > 1$ there is a positive probability of survival. $1 - q^N$, where N is the initial number of males, applying the fundamental theorem, the number q is a fixed point of g , that is a number such that $g(q) = q$ and $0 < q < 1$. the fixed point of g are found by solving the following equation;

$$\frac{1}{2} + \frac{q}{5-3q} = q$$

There are two solutions $q = 1$ and $q = \frac{5}{6}$ but only one solution satisfies $0 < q < 1$, namely $q = \frac{5}{6}$ and it should be noted that $q = 1$ will

always be one solution to $g(q) = q$ due to one of the properties of the probability generating function, $g(1) = 1$

To address the question about the probability of survival of family, it follows that one male has a probability of $\frac{5}{6}$ that is his line of descent becomes extinct and a probability of $\frac{1}{6}$, that his descendants will continue forever. The idea that the families die out is originated in theory and much has since been written about this problem. This work is concerned mainly with the statistical evidence of the extinction of families.

METHODOLOGY

Let us assume that an individual produces a random number ξ of offspring at a given time and produce no further offspring. In turn these descendants each produce further descendants at the next subsequent time at the same chance and let

$$P(\xi = k) = P_k \text{ where } k = 0, 1, 2, \dots \quad (1)$$

be the probability mass function of the number of offsprings generated by each individual. If each individual give rise to offspring independently, then ξ is independently identically distributed (iid) for each individual within a generation time n. In the nth generation, the X_n independently produce other $\xi_1^{(n)}, \xi_2^{(n)}, \dots, \xi_k^{(n)}$ and thus the total number of the individual produced in the (n+1)th generation is

$$X_{(n+1)} = \xi_1^{(n)} + \xi_2^{(n)} + \dots + \xi_k^{(n)} \dots \quad (2)$$

And this makeup the fundamental branching process equation.

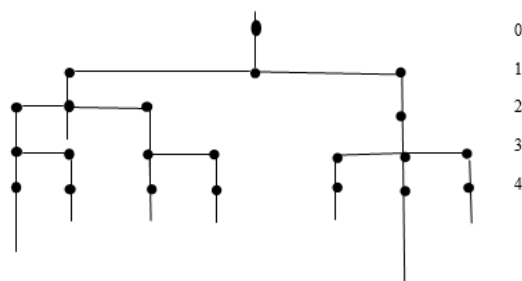


Fig 1.0 Picture of the branching process

Fig.1.0 above illustrates the general growth in branching process. A dot indicates a birth of an individual $X_0 = 1$ implies that there is one

individual in the initial generation, $X_1 = 3$ which produces three (3) individuals in the next generation. Thus it continues indefinitely or until extinction.

Probability of Extinction

Let $P[X_n = 0] = q_n$ Assume $P_0 > 0$

We are interested in knowing when the population would eventually die out, that is

$\lim_{n \rightarrow \infty} q_n$ If q_n is non-decreasing, $\lim_{n \rightarrow \infty} q_n = \pi$ exists and $0 < \pi \leq 1$.

Moreover, π is the smallest positive root of the equation $\varphi(z) = z$ ($0 < z < 1$)

It follows that

$$q_0 = 0 \text{ since } P(X_0 = 1) = 1 \text{ (by assumption)}$$

$$\varphi(z) = \sum_{k=0}^{\infty} P(X_n = k) Z^k \text{ and } P(X_n = 0) = q_n$$

Since

$$\varphi_{n+1}(0) = \varphi(\varphi_n(0)) \text{ and } q_{n+1} = \varphi q_n^n = 0, 1, 2, \dots \quad (3)$$

Then

$$q_1 = \varphi(q_0) > 0 = \varphi(0) = P_0 \text{ (by assumption)}$$

Now

$$q_0 = 0 \text{ and hence } q_1 > q_0 \text{ Suppose } q_{n-1} < q_n \text{ then}$$

Since $\varphi(z)$ is increasing in z ($0 < z \leq 1$)

We have

$$\varphi(q_{n-1}) \leq \varphi(q_n) \dots \quad (4)$$

Or $q_n \leq q_{n+1}$ by (3) and (4) hence q_n is non-decreasing in n

$$q_{n+1} = \varphi(q_n) \leq \varphi(1) \forall n = 1, 2, 3, \dots$$

Therefore

$$\lim_{n \rightarrow \infty} q_n = \pi \text{ (say) exists}$$

Now

$q_1 > 0$ and $q_n > 0 \forall n$ and $\{q_n\}$ is non-decreasing and $\pi > 0$.

We have

$$q_{n+1} = \varphi(q_n) \text{ taking the } \lim_{n \rightarrow \infty}$$

on both sides we get

$$\begin{aligned} \pi &= \lim_{n \rightarrow \infty} q_{n+1} \\ &= \lim_{n \rightarrow \infty} \varphi(q_n) \text{ (}\varphi(z)\text{ is continuous)} \\ &= \varphi(\pi) \end{aligned}$$

Now $\varphi(z) = z \rightarrow \pi$ is positive and hence π is positive root of $\varphi(z) = z$

Assume that S_0 is another positive square root of $\varphi(z) = z$

Then $q_0 = 0 < S_0$ assume $q_n < S_0$ and which implies

$$q_{n+1} \leq S_0 q_{n+1} = \varphi(q_n) \leq \varphi(S_0) = S_0 \dots (5)$$

$$\text{Now } \varphi_n(0) = P(X_n = 0) = q_n \leq 1$$

By induction on $n, q < S_0$ for $n = 0, 1, 2, \dots$

$$\text{Hence } \lim_{n \rightarrow \infty} q_n = \pi \leq S_0$$

Therefore π is the smallest positive root of $\varphi(z) = z$

To show that $\varphi(z) = z$ has a real root on $0 < z < 1$ iff $\mu > 1$

$\varphi(z) = z$ has only $z = 1$ as root in $0 < z < 1$ iff $\mu \leq 1$

It is known that the generating function $\frac{\varphi(z)}{z} = \frac{p_0}{z^2} + P_1 + P_2z + P_3z^2 + \dots + (0 < z < 1)$

Since this is a power series which is uniformly convergent

$$\begin{aligned} f'(z) &= -\frac{p_0}{z^2} + P_2 + 2P_3z + \\ \text{and } f''(z) &= \frac{2p_0}{z^3} + 2P_3 \dots (0 < z < 1) \\ \Rightarrow f'(z) \uparrow \text{ and } f'(0+) &= -\infty \dots (6) \end{aligned}$$

Case 1 $\mu > 1$

$$\begin{aligned} f'(1) &= -P_0 + P_2 + 2P_3 + \dots + \\ f'(0) &= P_1 + 2P_2 + 3P_3 + \dots + \\ &= \varphi(1) > 0 \end{aligned}$$

Thus there exists $0 < \beta < 1$, such that $f'(\beta) = 0$ and which implies that $f'(z) < 0$ for $0 < z < \beta$. Hence $f'(z) \downarrow$ in z for $0 < z < \beta$ and $f'(z) \uparrow$ in z for $\beta < z < 1$

Now

$$\begin{aligned} f(1) &= \frac{\varphi(1)}{1} = \frac{1}{1} = 1 \\ f(0+) &= \infty \dots (7) \end{aligned}$$

Therefore there exists $0 < a < 1$ such that $f(a) = 1$ that is $\varphi(a) = a$

Case 2: $\mu \leq 1$

Since $\beta = 1$ hence $f'(z) < 0$ in z for $0 < z < 1$

$$\text{For } f(2) \downarrow f(0+) = \infty f(1) = 1$$

Hence, there does not exist $0 < a < 1$ such that $f(a) = 1$ and hence $z = 1$ is the only root of $\varphi(z) = z$

Method 1

$$\begin{aligned} P(\xi = 0) &= v > 0, P(\xi = 2) = \lambda \\ &> 0, P(\xi = 1) &= 1 - v - \lambda \\ &> 0 \end{aligned}$$

$$\text{Then } \varphi(z) = v + (1 - v - \lambda)z + \lambda z^2$$

To find the solution of $\varphi(z) = z$

We have

$$\begin{aligned} v + (1 - v - \lambda)z + \lambda z^2 &= z \\ v + z - vz - \lambda z + \lambda z^2 &= z \\ v - vz - \lambda z + \lambda z^2 &= z - z \\ v - (v - \lambda)z + \lambda z^2 &= 0 \end{aligned}$$

$$\text{Thus } (z - 1)(\lambda z - v) = 0 \Rightarrow z = 1 \text{ or } z = \frac{v}{\lambda}$$

$$\text{Then } \mu = \varphi'(1) = 1 - v - \lambda + 2\lambda = 1 - v + \lambda$$

Therefore

$$\mu > 1 \Rightarrow 1 - v + \lambda > 1 \text{ or } -v + \lambda > 0 \text{ or } \lambda > v$$

Thus if $\mu > 1$, that is $\lambda > v$, the population dies out with probability of $\pi = \frac{v}{\lambda} < 1$ and if $\mu \leq 1$

That is $\lambda \leq v$, that is $\frac{v}{\lambda} > 1$

Then $\pi = 1$ that is, the population is to die out (Athreya & Ney, 1972)

If $\mu > 1$ then with the probability π the population become extinct and with probability $1 - \pi$ the population explodes

If $\mu \leq 1$ then with probability 1, the population becomes extinct [regardless of the actual value of the mean $E(X_1) = \mu > 1$, the probability that the n -th generation will consist of any positive finite number of individuals tend to zero as $n \rightarrow \infty$, with the probability of extinction tending to π . in this circumstance, we say that $X_n \rightarrow \infty$ as $n \rightarrow \infty$ with probability $1 - \pi$] (Bruss, 1984)

Let T be the time till extinction (first passage time to state zero)

Then $[T = n] = [X_n = 0, X_{n-1} \neq 0]$

Hence $P[T = n] = q_n - q_{n-1} = P[X_n = 0] - P[X_{n-1} = 0]$

Hence $E(T) = \sum_{n=1}^{\infty} n(q_n - q_{n-1}) = q_1 - q_0 + 2q_2 - 2q_1 + 3q_3 - 3q_2 + \dots = -q_0 - q_1 - q_2 - q_3 - \dots < 0$

This is because the series $\sum_{n=1}^{\infty} n(q_n - q_{n-1})$ is convergent in the rearrangement of the series.

Now, $\lim_{n \rightarrow \infty} q_n = \pi > 0 \Rightarrow q_n = d > 0$ therefore $\sum nq_n$ is divergent (Lyons & Peres, 2016)

Method 2

$\varphi(z) = P_0 + P_1Z, 0 < P_0 < 1$ and $P_0 + P_1 = 1$ then

$\varphi_2(z) = P_0 + P_1(P_0 + P_1Z) = P_0 + P_0P_1 + P_1^2Z$

thus

$\varphi_n(z) = P_0 + P_0P_1^2 + \dots + P_0P_1^{n-1} + P_1^nZ$ (By iteration)

$= P_0 \frac{1 - P_1^n}{1 - P_1} + P_1^nZ$

$= P_0 \frac{1 - P_1^n}{P_0} + P_1^nZ$

$= (1 - P_1^n) + P_1^nZ$

Hence

$q_n = P(X_n = 0) = 1 - P_1^n, \rightarrow 1$ as $n \rightarrow \infty$

Also $\phi(z) = Z = P_0 + P_1Z = Z(1 - P_1) = P_0 \Rightarrow Z = 1$

That is population will surely die out

Here

$P(T = n) = q_n - q_{n-1} = 1 - P_1^n - 1 + P_1^{n-1} = P_0P_1^{n-1}$

that is, the population is alive with geometric distribution:

$(T) = \sum_{n=1}^{\infty} nP_0P_1^{n-1} = P_0 = P_0 \frac{1}{(1-P_1)^2} = \frac{P_0}{(P_0)^2} = \frac{1}{P_0}$

The total progeny is $Y_{\infty} = \sum_{n=0}^{\infty} Z_n$ from the extinction probability $P(Y_{\infty} < \infty) = q$

And putting

$Y_n = \sum_{k=0}^n Z_k$

We obtained,

$(Y_n) = \frac{1 - \mu^{n+1}}{1 - \mu}$ if $\mu \neq 1$

And if $\mu = 1$

$E(Y_n) = n + 1$

If $P_{jk} = P(Z_{n+1} = k | Z_n = j)$, then verily, it can be shown that for any $j \leq k$

$P(Y_{\infty} = k) = \frac{j}{k} P_{K,k-j}$

RESULTS AND DISCUSSION

Illustration 1

Consider a parent (male) that can produce at most two male offspring during his life time. We only consider male offspring based on the fact that family names are maintained by sons only. The probabilities of the male offspring produced are as follows:

Probability of having no male is: $P_0 = 0.1$

Probability of having one male is: $P_1 = 0.6$

Probability of having two males is: $P_2 = 0.3$

and $\sum_{k=0}^{\infty} P_k = P_0 + P_1 + P_2 \Rightarrow 0.1 + 0.6 + 0.3 = 1$

The probability of extinction in each generation is given by

$q_n = P_0 + P_1q_{n-1} + P_2(q_{n-1})^2$

With $q = 0$, here the extinction probability is calculated from generation one (X_1) to generation 30 (X_{30})

$P(X_1) = 0.1 + 0.6(0) + 0.3(0)^2 = 0.1$

$P(X_2) = 0.1 + 0.6(0.1) + 0.3(0.1)^2 = 0.163$

$P(X_3) = 0.1 + 0.6(0.163) + 0.3(0.163)^2 = 0.2058$

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.. .. .

$P(X_{28}) = 0.1 + 0.6(0.3328) + 0.3(0.3328)^2 = 0.3329$

$P(X_{29}) = 0.1 + 0.6(0.3329) + 0.3(0.3329)^2 = 0.3330$

$P(X_{30}) = 0.1 + 0.6(0.3330) + 0.3(0.3330)^2 = 0.3$

Table 1

| Genera- tion | Probability of extinction | Genera- tion | Probability of extinction |
|-----------------|------------------------------|-----------------|---------------------------------|
| 1 | 0.1 | 16 | 0.3277 |
| 2 | 0.163 | 17 | 0.3288 |
| 3 | 0.2058 | 18 | 0.3297 |
| 4 | 0.2063 | 19 | 0.3304 |
| 5 | 0.2584 | 20 | 0.331 |
| 6 | 0.2751 | 21 | 0.3315 |
| 7 | 0.2878 | 22 | 0.319 |
| 8 | 0.2975 | 23 | 0.3322 |
| 9 | 0.3053 | 24 | 0.3324 |
| 10 | 0.3111 | 25 | 0.3326 |
| 11 | 0.3157 | 26 | 0.3327 |
| 12 | 0.3193 | 27 | 0.3328 |
| 13 | 0.3222 | 28 | 0.3329 |
| 14 | 0.3245 | 29 | 0.3330 |
| 15 | 0.3263 | 30 | 0.3331 |

Hence, the probability of extinction q_n is non-decreasing and as n increases, it tends to one (1) or the probability of extinction tends to the ultimate probability of extinction as time goes on. It can be concluded that all families are to die out as stated by the earlier pioneers of branching process. (Karlin & Kaplan, 1973; Hull, 1982; Alsmeyer & Osler, 2002; Gonz_alez, Molina & Mota 2000)

$$n \rightarrow \infty \quad q_n = 1$$

Illustration 2

From example 1, suppose we let the probability for the number of offspring to be as follows:

Probability of having no son (male offspring) is $P_0 = 0.2$

Probability of having one male offspring is $P_1 = 0.5$

Probability of having two male offspring is $P_2 = 0.3$

And $\sum_{k=0}^2 p_k = 0.2 + 0.5 + 0.3 = 1$

Then the probability of extinction would be given by $q_n = P_0 + P_1q_{n-1} + P_2+(q_{(n-1)})^2$

With $q_0 = 0$.

In this case we would consider calculating the probability of extinction from generation one (X_1) to generation 10 (X_{10})

$$P(X_1) = 0.2 + 0z. 3(0) + 0.5(0)^2 = 0.2$$

$$P(X_2) = 0.2 + 0.3(0.2) + 0.5(0.2)^2 = 0.28$$

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.. .. .

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$$P(X_{17}) = 0.2+0.3(0.3994) + 0.5(0.3994)^2 = 0.3996$$

$$P(X_{18}) = 0.2+0.3(0.3996) + 0.5(0.3996)^2 = 0.3997$$

$$P(X_{19}) = 0.2+0.3(0.3997) + 0.5(0.3997)^2 = 0.3998$$

$$P(X_{20}) = 0.2+0.3(0.3998) + 0.5(0.3998)^2 = 0.3999$$

Table 2

| Genera- tion | Probability of extinction | Genera- tion | Probability of extinction |
|-----------------|---------------------------------|-----------------|---------------------------------|
| 1 | 0.2 | 11 | 0.3962 |
| 2 | 0.28 | 12 | 0.3973 |
| 3 | 0.3232 | 13 | 0.3981 |
| 4 | 0.3492 | 14 | 0.3987 |
| 5 | 0.3657 | 15 | 0.3991 |
| 6 | 0.3766 | 16 | 0.3994 |
| 7 | 0.3839 | 17 | 0.3996 |
| 8 | 0.3889 | 18 | 0.3997 |
| 9 | 0.3923 | 19 | 0.3998 |
| 10 | 0.3946 | 20 | 0.3999 |

Thus by inspection, the probability of extinction q_n is non-decreasing and as n increases it tends to 1 as time goes on. It follows that all family names are to die out in the long run.

Conclusion

A model of this sort is found useful in many real life problems. The probability of extinction depends extensively on the value of μ which ensures that the branching process will die out with probability 1. There are values of μ that guarantee that the probability of extinction will be strictly less than 1. If the mean number of offspring per individual is more than 1 (that is, on average, if individuals replace themselves with a bit extra), then the branching process is not guaranteed to die out. However, if the mean number of offspring per individual μ is 1 or less, the process is guaranteed to become extinct.

Based on the outcome of this research, it is established that non-deterministic real life problems can be tackled using branching process. Branching process is thus recommended to serve as a suitable tool in solving problems such as: cell division, family extinction and nuclear chain reaction. It may be used to fit appropriate models for various real life problems.

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